Mental Math Tricks

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compute

(1) verb: to determine by calculation; reckon; calculate

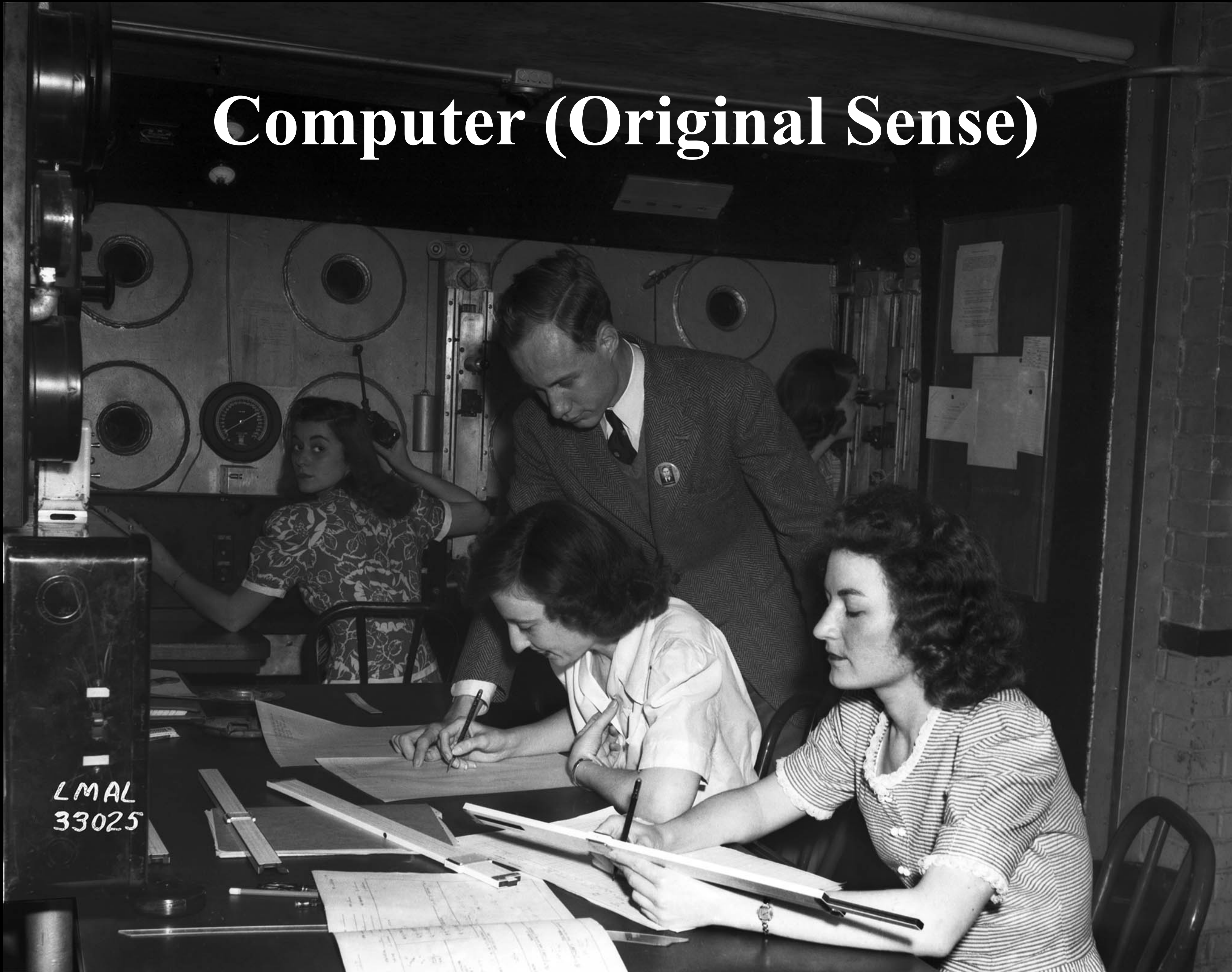
computer

(1) noun: a programmable electronic device designed to accept data, perform prescribed mathematical and logical operations at high speed, and display the results of these operations.

computer

(2) noun: a person who computes; computist.

Computer (Original Sense)



Why Study This?

- Awesome

- Better understanding of numbers

- Occasionally faster than busting out the calculator

Mental Calculation

(1) Brute force

(2) Stupid number tricks

(3) Approximations

Brute Force

- Based on realization that calculation is not *that* hard
- Just compute it already!
- Faster with practice!

Clever Tricks
Demo: Squaring Numbers

- Tell me a 2-digit number (e.g. 35)
- I tell you the square ($35^2 = 1{,}225$)!

(p≈0.2 I'll screw up . . .)

Clever Tricks
Squaring Numbers

Works by a simple algebra trick:

$$x^2 = (x-a)(x+a) + a^2$$

Clever Tricks
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$$35^2 = (35-a)(35+a) + a^2$$

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Squaring Numbers

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$$x^2 = (x-a)(x+a) + a^2$$

$$35^2 = (35-a)(35+a) + a^2$$

$$35^2 = (35-5)(35+5) + 5^2$$

Recursive!

Clever Tricks
Squaring Numbers

Works by a simple algebra trick:

$$x^2 = (x-a)(x+a) + a^2$$

$$35^2 = (35-a)(35+a) + a^2$$

$$35^2 = (35-5)(35+5) + 5^2$$

$$35^2 = (30)(40) + 25$$

Clever Tricks
Squaring Numbers

Works by a simple algebra trick:

$$x^2 = (x-a)(x+a) + a^2$$

$$35^2 = (35-a)(35+a) + a^2$$

$$35^2 = (35-5)(35+5) + 5^2$$

$$35^2 = (30)(40) + 25$$

$$35^2 = 1200 + 25$$

Clever Tricks
Squaring Numbers

Works by a simple algebra trick:

$$x^2 = (x-a)(x+a) + a^2$$

$$35^2 = (35-a)(35+a) + a^2$$

$$35^2 = (35-5)(35+5) + 5^2$$

$$35^2 = (30)(40) + 25$$

$$35^2 = 1200 + 25$$

$$35^2 = 1225$$

Clever Tricks
Squaring Numbers

- Also possible with larger numbers
  - (harder and more-error-prone)

- Certain numbers make it easier
  - E.g. Last two digits less than ~30
  - E.g. Rest of the digits < ~5
  - Last digit being 5 or 0

Clever Tricks
Demo: Cube Roots

- Take a 2-digit number (e.g. 35)
- Cube it ($35 \cdot 35 \cdot 35 = 35^3 = 42{,}875$)
- Tell me the cube (42,875)
- I tell you the cube root (35)!

Clever Tricks
Cube Roots

- Works via two lookups into a small table:

```
 0³ =    0
 1³ =    1
 2³ =    8
 3³ =   27
 4³ =   64
 5³ =  125
 6³ =  216
 7³ =  343
 8³ =  512
 9³ =  729
10³ = 1000
```

Clever Tricks
Cube Roots

- Works via two lookups into a small table:

0^3	=	0
1^3	=	1
2^3	=	8
3^3	=	27
4^3	=	64
5^3	=	125
6^3	=	216
7^3	=	343
8^3	=	512
9^3	=	729
10^3	=	1000

42,875

Clever Tricks
Cube Roots

- Works via two lookups into a small table:

0^3	=	0
1^3	=	1
2^3	=	8
3^3	=	27
4^3	=	64
5^3	=	125
6^3	=	216
7^3	=	343
8^3	=	512
9^3	=	729
10^3	=	1000

42,875

42 is larger than 27
and less than 64

Clever Tricks
Cube Roots

- Works via two lookups into a small table:

0^3	=	0
1^3	=	1
2^3	=	8
3^3	=	27
4^3	=	64
5^3	=	125
6^3	=	216
7^3	=	343
8^3	=	512
9^3	=	729
10^3	=	1000

42,875

42 is larger than 27
and less than 64, so the
answer must start with 3.

3_

Clever Tricks
Cube Roots

- Works via two lookups into a small table:

0^3	=	0
1^3	=	1
2^3	=	8
3^3	=	27
4^3	=	64
5^3	=	125
6^3	=	216
7^3	=	343
8^3	=	512
9^3	=	729
10^3	=	1000

42,875

3_

Clever Tricks
Cube Roots

- Works via two lookups into a small table:

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$1^3 = 1$
$2^3 = 8$
$3^3 = 27$
$4^3 = 64$
$5^3 = 125$
$6^3 = 216$
$7^3 = 343$
$8^3 = 512$
$9^3 = 729$
$10^3 = 1000$

42,875

The last digit of the cube matches the last digit of 125

3_

Clever Tricks
Cube Roots

- Works via two lookups into a small table:

0^3	=	0
1^3	=	1
2^3	=	8
3^3	=	27
4^3	=	64
5^3	=	125
6^3	=	216
7^3	=	343
8^3	=	512
9^3	=	729
10^3	=	1000

42,875

The last digit of the cube matches the last digit of 125, so the second digit is 5.

35

Clever Tricks
Cube Roots

- Works via two lookups into a small table:

0^3 =	0
1^3 =	1
2^3 =	8
3^3 =	27
4^3 =	64
5^3 =	125
6^3 =	216
7^3 =	343
8^3 =	512
9^3 =	729
10^3 =	1000

42,875

35

Clever Tricks
Cube Roots

- Can be extended to 3-digit cube roots (a bit nasty):

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16,387,064

Clever Tricks
Cube Roots

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16,387,064

First digit computed same as last time:
$2^3=8 \leq 16 < 3^3=27$

2 _ _

Clever Tricks
Cube Roots

- Can be extended to 3-digit cube roots (a bit nasty):

16,387,064

Last digit computed same as last time: 4 is the ending of $4^3=64$

2_4

Clever Tricks
Cube Roots

- Can be extended to 3-digit cube roots (a bit nasty):

16,387,064

1. Compute mod 11 of the cube (add and subtract digits, then mod, is easier)

+4 –6 +0 –7 +8 –3 +6 –1 = 1

2_4

Clever Tricks
Cube Roots

- Can be extended to 3-digit cube roots (a bit nasty):

16,387,064

1. Compute mod 11 of the cube (add and subtract digits, then mod, is easier)

+4 –6 +0 –7 +8 –3 +6 –1 = 1

2. Find the cube root mod 11

1^3 mod 11 = 1

0^3 mod 11 =	0
1^3 mod 11 =	1
2^3 mod 11 =	8
3^3 mod 11 =	5
4^3 mod 11 =	9
5^3 mod 11 =	4
6^3 mod 11 =	7
7^3 mod 11 =	2
8^3 mod 11 =	6
9^3 mod 11 =	3
10^3 mod 11 =	10

2 _ 4

Clever Tricks
Cube Roots

- Can be extended to 3-digit cube roots (a bit nasty):

16,387,064

1. Compute mod 11 of the cube (add and subtract digits, then mod, is easier)
   $+4 - 6 + 0 - 7 + 8 - 3 + 6 - 1 = 1$
2. Find the cube root mod 11
   $1^3 \bmod 11 = 1$
3. Compute sum of the first and last digits of the cube root.
   $2 + 4 = 6$

Clever Tricks
Cube Roots

- Can be extended to 3-digit cube roots (a bit nasty):

16,387,064

1. Compute mod 11 of the cube (add and subtract digits, then mod, is easier)

   +4 –6 +0 –7 +8 –3 +6 –1 = 1

2. Find the cube root mod 11

   1^3 mod 11 = 1

3. Compute sum of the first and last digits of the cube root.

   2 + 4 = 6

4. Subtract and mod by 11.

   6 – 1 = 5

2 _ 4

Clever Tricks
Cube Roots

- Can be extended to 3-digit cube roots (a bit nasty):

16,387,064

1. Compute mod 11 of the cube (add and subtract digits, then mod, is easier) — $+4 - 6 + 0 - 7 + 8 - 3 + 6 - 1 = 1$
2. Find the cube root mod 11 — $1^3 \bmod 11 = 1$
3. Compute sum of the first and last digits of the cube root. — $2 + 4 = 6$
4. Subtract and mod by 11. — $6 - 1 = 5$

254

Clever Tricks
Cube Roots

- Can be extended to 3-digit cube roots (a bit nasty):

16,387,064

254

Clever Tricks
Day of the Week

- Compute day of the week for any day in history!
  - Lengthy (albeit simple) calculations
  - Some memorization
  - Check your answer: https://pastebin.com/eGWj0TNK

Clever Tricks
Day of the Week

November 9, 2018 is a __________

Clever Tricks
Day of the Week

November 9, 2018 is a __________

Step 1: Is 2018 a leap year?

No (leap years are divisible by 4, with the exception of those divisible by 100 but not 400.)

Clever Tricks
Day of the Week

November 9, 2018 is a __________

Step 2: Calculate offset due to year

```
T = 18 (year mod 100)
T = 18 (if T is odd, add 11)
T =  9 (divide T by 2)
T = 20 (if T is odd, add 11)
T =  6 (mod 7)
```

Clever Tricks
Day of the Week

November 9, 2018 is a __________

T = 6 (from step 2)

Step 3: Add century's first Sunday

0	2	4	5
n/a	n/a	1500	1600
1700	1800	1900	2000
2100	2200	2300	etc.

Clever Tricks
Day of the Week

November 9, 2018 is a __________

T = 6 (from step 2)

Step 3: Add century's first Sunday

0	2	4	5
n/a	n/a	1500	1600
1700	1800	1900	2000
2100	2200	2300	etc.

Clever Tricks
Day of the Week

November 9, 2018 is a __________

T = 6 (from step 2)

Step 3: Add century's first Sunday
T = 11 (add 5 from table)

0	2	4	5
n/a	n/a	1500	1600
1700	1800	1900	2000
2100	2200	2300	etc.

Clever Tricks
Day of the Week

November 9, 2018 is a ___________

T = 6 (from step 2)

Step 3: Add century's first Sunday

T = 11 (add 5 from table)

T = 4 (mod 7)

0	2	4	5
n/a	n/a	1500	1600
1700	1800	1900	2000
2100	2200	2300	etc.

Clever Tricks
Day of the Week

November 9, 2018 is a _________

`T = 4` (from step 3)

Step 4: Add month's doomsday

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	
1	2	3	4	5	6	7	8	9	10	11	12	(month number)
3	0	0	4	9	6	11	8	5	10	7	12	(ordinary years)
4	1	0	4	9	6	11	8	5	10	7	12	(leap years)

Clever Tricks
Day of the Week

November 9, 2018 is a __________

T = 4 (from step 3)

Step 4: Add month's doomsday

Not *too* hard to memorize . . .

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	
1	2	3	4	5	6	7	8	9	10	11	12	(month number)
3	0	0	4	9	6	11	8	5	10	7	12	(ordinary years)
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Clever Tricks
Day of the Week

November 9, 2018 is a __________

T = 4 (from step 3)

Step 4: Add month’s doomsday

Not *too* hard to memorize . . .

Even-numbered months are just their numbers
(except Feb, which is just `int(is_leapyear)`)

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	
1	2	3	4	5	6	7	8	9	10	11	12	(month number)
3	0	0	4	9	6	11	8	5	10	7	12	(ordinary years)
4	1	0	4	9	6	11	8	5	10	7	12	(leap years)

Clever Tricks
Day of the Week

November 9, 2018 is a __________

T = 4 (from step 3)

Step 4: Add month's doomsday

Not *too* hard to memorize . . .

(9 and 5) and (7 and 11) form a pair that swap

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	
1	2	3	4	5	6	7	8	9	10	11	12	(month number)
3	0	0	4	9	6	11	8	5	10	7	12	(ordinary years)
4	1	0	4	9	6	11	8	5	10	7	12	(leap years)

Clever Tricks
Day of the Week

November 9, 2018 is a __________

T = 4 (from step 3)

Step 4: Add month's doomsday

Not *too* hard to memorize . . .

All that's left is Jan and Mar

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	
1	2	3	4	5	6	7	8	9	10	11	12	(month number)
3	0	0	4	9	6	11	8	5	10	7	12	(ordinary years)
4	1	0	4	9	6	11	8	5	10	7	12	(leap years)

Clever Tricks
Day of the Week

November 9, 2018 is a ________

T = 4 (from step 3)

Step 4: Add month's doomsday

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	
1	2	3	4	5	6	7	8	9	10	11	12	(month number)
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4	1	0	4	9	6	11	8	5	10	7	12	(leap years)

Clever Tricks
Day of the Week

November 9, 2018 is a _________

T = 4 (from step 3)

Step 4: Add month's doomsday

T = 11 (add 7 from table)

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	
1	2	3	4	5	6	7	8	9	10	11	12	(month number)
3	0	0	4	9	6	11	8	5	10	7	12	(ordinary years)
4	1	0	4	9	6	11	8	5	10	7	12	(leap years)

Clever Tricks
Day of the Week

November 9, 2018 is a __________

T = 4 (from step 3)

Step 4: Add month's doomsday

T = 11 (add 7 from table)

T = 4 (mod 7)

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	
1	2	3	4	5	6	7	8	9	10	11	12	(month number)
3	0	0	4	9	6	11	8	5	10	7	12	(ordinary years)
4	1	0	4	9	6	11	8	5	10	7	12	(leap years)

Clever Tricks
Day of the Week

November 9, 2018 is a __________

T = 4 (from step 4)

Step 5: Subtract from day

Clever Tricks
Day of the Week

November 9, 2018 is a __________

T = 4 (from step 4)

Step 5: Subtract from day

T = 5 (9 – 4)

Clever Tricks
Day of the Week

November 9, 2018 is a __________

T = 4 (from step 4)

Step 5: Subtract from day

T = 5 (9 – 4)

Step 6: Convert to weekday

Sun	Mon	Tues	Wed	Thurs	Fri	Sat
0	1	2	3	4	5	6

Clever Tricks
Day of the Week

November 9, 2018 is a __________

T = 4 (from step 4)

Step 5: Subtract from day

T = 5 (9 – 4)

Step 6: Convert to weekday

Sun	Mon	Tues	Wed	Thurs	Fri	Sat
0	1	2	3	4	5	6

Clever Tricks
Day of the Week

November 9, 2018 is a Friday!

T = 4 (from step 4)

Step 5: Subtract from day

T = 5 (9 – 4)

Step 6: Convert to weekday

Friday (lookup T=5 in zero-indexed week)

Sun	Mon	Tues	Wed	Thurs	Fri	Sat
0	1	2	3	4	5	6

Clever Tricks
Day of the Week

November 9, 2018 is a Friday!

- If using Julian calendar (our current Gregorian calendar was invented 1582 in Italy and adopted slowly), rules are a little simpler.
- Gets easier with practice, I promise.
  - (Though still only useful if you don't have a calendar)

People Who Are Famous At This

- Gauss. E.g.: “Compute 1+2+...+100”
- Zerah Colburn. E.g.:
  - Q: “How many seconds are in 11 years?”
  - A: “346,896,000” (answered in 4 seconds, correct neglecting leap years)
- George Parker Bidder. E.g.:
  - Q: “sqrt(119,550,669,121) = ?”
  - A: “345,761” (answered in 30 seconds)
- Shakuntala Devi. E.g. (video), and:
  - Q: “7,686,369,774,870 × 2,465,099,745,779 = ?”
  - A: “18,947,668,177,995,426,462,773,730” (answered in 20 seconds, allegedly)
- Alexander Craig Aitken. E.g.: 5×5 multiplication near-instantly
- Thomas Fuller. E.g.:
  - Q: “$7^8 \times 6$”
  - A: “34,588,806” (answered in < 10 minutes)
- Arthur Benjamin E.g. (video)

More Topics to Return To . . .

- Better: Left-to-right addition/subtraction
- Faster: "Criss-cross" multiplication
- Stronger: Mnemonics
- Trick: Multiplying by 11, square roots?
- Tweak: Faster x-a in square trick by doubling, generalization to close-together multiplication
- Perceive: divisibility rules
- Guess: square roots, exponents

Questions / Discussion

Picture Credits

- http://crgis.ndc.nasa.gov/crgis/images/3/30/L-33025.jpg
- https://upload.wikimedia.org/wikipedia/commons/thumb/8/8f/Slide_rule_example2_with_labels.svg/550px-Slide_rule_example2_with_labels.svg.png